

Quantitative estimates for the effect of disorder on low-dimensional lattice systems

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Lattice systems with compact state space

- We discuss **statistical physics systems on \mathbb{Z}^d** , aiming to develop a **quantitative understanding** of the effect of adding **disorder** to them.
- We start with the case of a **compact state space**.
- **Setup**: (1) Compact metric space S equipped with a Borel measure κ .
(2) Translation-invariant **finite range and finite energy Hamiltonian H** .
- As usual, for a finite domain $\Lambda \subset \mathbb{Z}^d$, at temperature T and with boundary conditions $\tau: \mathbb{Z}^d \rightarrow S$, configurations $\sigma: \mathbb{Z}^d \rightarrow S$ coinciding with τ outside Λ are sampled from the probability measure with density

$$\frac{1}{Z_{T,\Lambda,\tau}} \exp\left(-\frac{1}{T} H_\Lambda(\sigma)\right)$$

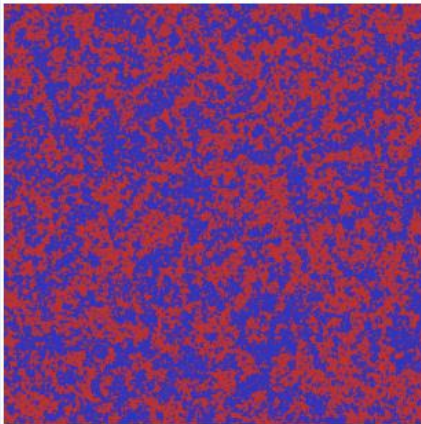
with respect to the measure $\prod_v d\kappa(\sigma_v)$, where $Z_{T,\Lambda,\tau}$ is the **partition function** and H_Λ contains the terms in the Hamiltonian depending on the spins in Λ .

Periodic boundary conditions and the **zero-temperature limit** are also allowed.

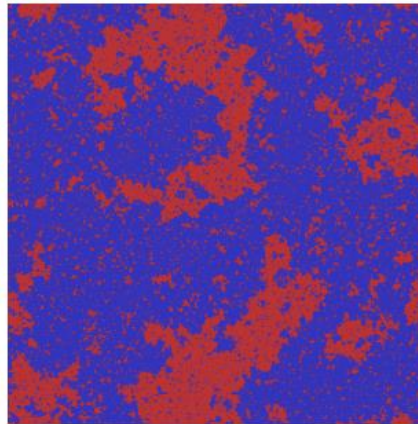
- **Examples**: **Ising model**: $S = \{-1, 1\}$, $\kappa = \text{counting}$, $H(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v$
- **Potts model**: $S = \{1, 2, \dots, q\}$, $\kappa = \text{counting}$, $H(\sigma) = -\sum_{u \sim v} 1_{\sigma_u = \sigma_v}$
- **Spin $O(n)$ model with $n \geq 2$** : $S = \mathbb{S}^{n-1}$, $\kappa = \text{uniform}$, $H(\sigma) = \sum_{u \sim v} |\sigma_u - \sigma_v|^2$

Phase transitions in pure systems I

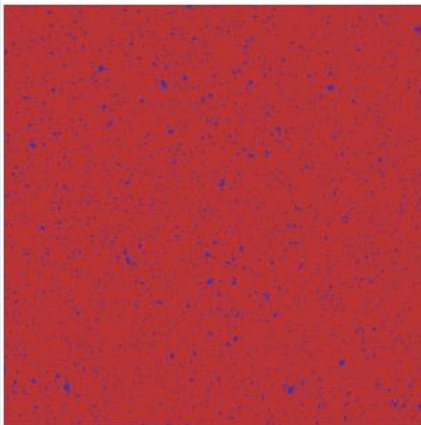
- **Ising model:** $S = \{-1, 1\}$, $\kappa = \text{counting}$, $H(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v$.
- The Ising model undergoes a phase transition in dimensions $d \geq 2$ as the temperature is lowered, from a disordered to an ordered state.
- Similar behavior for the q -state Potts model ($S = \{1, 2, \dots, q\}$, $H(\sigma) = -\sum_{u \sim v} 1_{\sigma_u = \sigma_v}$).



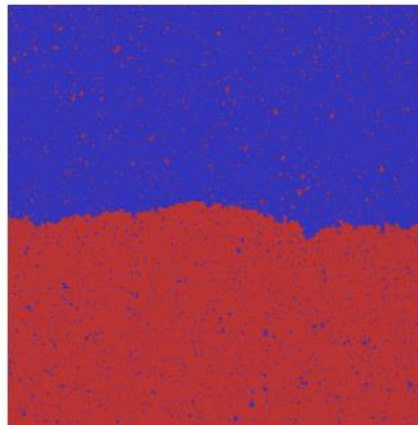
(a) $\beta = 0.4 < \beta_c$



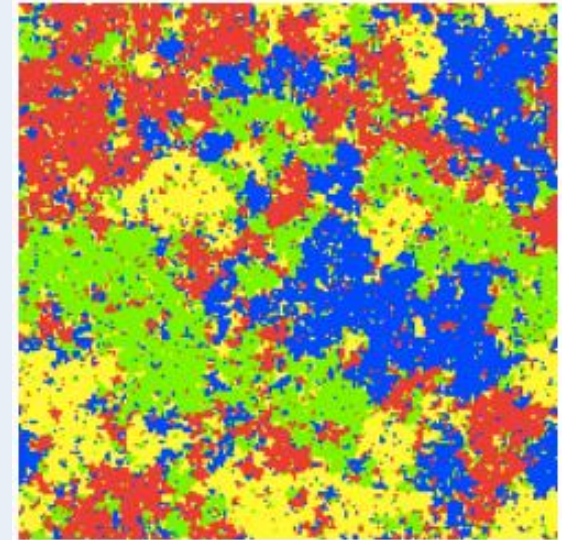
(b) $\beta = \beta_c \approx 0.4407$



(c) $\beta = 0.5 > \beta_c$



(d) $\beta = 0.5$ with Dobrushin boundary conditions

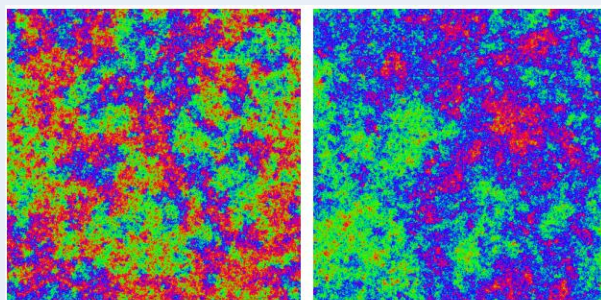


$d=2$ Potts model with $q=4$ at criticality.
Simulation by [Beffara](#).

Simulation from
[Spinka–Peled 2019](#)

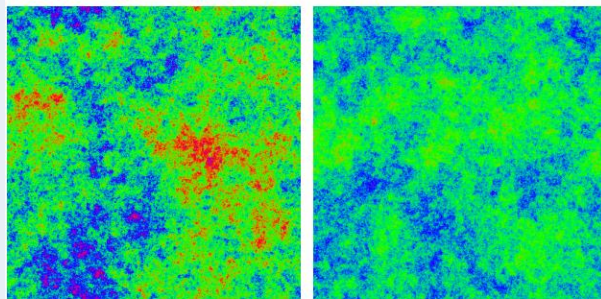
Phase transitions in pure systems II

- Spin $O(n)$ model with $n \geq 2$: $S = \mathbb{S}^{n-1}$, $\kappa = \text{uniform}$, $H(\sigma) = \sum_{u \sim v} |\sigma_u - \sigma_v|^2$
- Mermin–Wagner theorem: The spin $O(n)$ model does not exhibit an ordered phase in two dimensions (even at low temperature).
- Fröhlich–Simon–Spencer theorem: A low-temperature ordered phase exists in dimensions $d \geq 3$.
- In two dimensions:
 - $n=2$ (XY model): Berezinskii–Kosterlitz–Thouless transition (proof by Fröhlich–Spencer) from exponential to power-law decay of correlations as temperature is lowered.
 - $n=3$ (Heisenberg model): Polyakov conjecture – exponential decay at all temperatures.



(a) $\beta = 1$

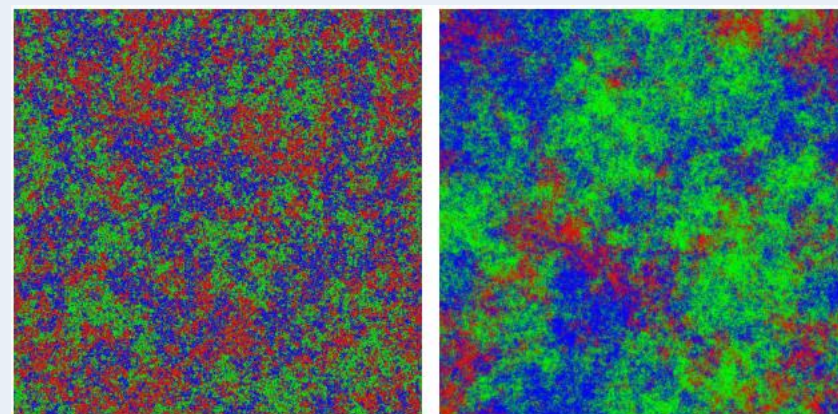
(b) $\beta = 1.12$



(c) $\beta = 1.5$

(d) $\beta = 3$

XY model simulation from
Spinka–Peled 2019



(a) $\beta = 2$

(b) $\beta = 10$

Heisenberg model simulation
from Spinka–Peled 2019

Disordered lattice systems

- **Noised observables:** Let $f: S^{\mathbb{Z}^d} \rightarrow \mathbb{R}^m$, for some $m \geq 1$, be a **bounded** measurable function depending on the spins in a **finite neighborhood of the origin**.

Disorder: Let $(\eta_v)_{v \in \mathbb{Z}^d}$ be independent standard m -dimensional Gaussian vectors.

Disordered Hamiltonian: $H^\eta(\sigma) = H(\sigma) - \lambda \sum_v \eta_v \cdot f(\mathcal{T}_v(\sigma))$

where $\mathcal{T}_v(\sigma)$ is the configuration σ translated by v .

- **Examples: Random-field Ising model:** $m = 1$ and $f(\sigma) = \sigma_0$. Thus

$$H^\eta(\sigma) = - \sum_{u \sim v} \sigma_u \sigma_v - \lambda \sum_v \eta_v \sigma_v$$

- **Edwards-Anderson spin glasses:** $S = \{-1, 1\}$, $\kappa = \text{counting}$, $f(\sigma) = (\sigma_{e_j} \sigma_0)_{j=1}^d$.

$$H^\eta(\sigma) = -\lambda \sum_{u \sim v} \eta_{u,v} \sigma_u \sigma_v$$

- **Random-field q -state Potts model:** $m = q$ and $f(\sigma) = (1_{\sigma_0=1}, \dots, 1_{\sigma_0=q})$. Thus

$$H^\eta(\sigma) = - \sum_{u \sim v} 1_{\sigma_u = \sigma_v} - \lambda \sum_v \sum_{k=1}^q \eta_{v,k} 1_{\sigma_v = k}$$

- **Random-field spin $O(n)$ model, $n \geq 2$:** $m = n$ and $f(\sigma) = \sigma_0$ (with $S^{n-1} \subset \mathbb{R}^n$),

$$H^\eta(\sigma) = \sum_{u \sim v} |\sigma_u - \sigma_v|^2 - \lambda \sum_v \eta_v \cdot \sigma_v$$

Imry-Ma phenomenon

- Imry-Ma (1975) considered the effects of disorder for the random-field Ising and spin $O(n)$ models, and predicted that in low dimensions, an arbitrarily small disorder strength λ causes the models to lose their ordered phase, as follows:
The random-field Ising model is disordered at all temperatures for $d \leq 2$.
The random-field spin $O(n)$ model is disordered at all temperatures for $d \leq 4$.
- Aizenman-Wehr (1989) proved the predictions as part of a general statement.
- **Notation:** Write $\Lambda_L := \{-L, \dots, L\}^d$. For each disorder η , write $\langle \cdot \rangle_\mu$ for the thermal expectation according to a Gibbs measure μ of the η -disordered system.
Write \mathbb{P} and \mathbb{E} for the probability and expectation operator over η .
- **Theorem** (Aizenman-Wehr, special case): For a disordered lattice system with compact state space (as discussed above) in dimensions $d = 1, 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, the limit

$$\lim_{L \rightarrow \infty} \frac{1}{L^d} \sum_{v \in \Lambda_L^d} \langle f(\mathcal{T}_v(\sigma)) \rangle_\mu$$

exists and has the same value for all Gibbs measures μ and almost all η .

The same holds in dimensions $1 \leq d \leq 4$ for the spin $O(n)$ models with $n \geq 2$.

- **Our goal:** Develop a quantitative understanding of this phenomenon.

Random-field Ising model

- Random-field Ising model Hamiltonian: $H^\eta(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v - \lambda \sum_v \eta_v \sigma_v$
- The disordered model still satisfies the usual **monotonicity (FKG) properties**. In particular, the model has maximal and minimal Gibbs measures $\mu^{\eta,+}$ and $\mu^{\eta,-}$, arising in the thermodynamic limit from constant boundary conditions. The Aizenman-Wehr theorem implies that $\mu^{\eta,+} = \mu^{\eta,-}$ in two dimensions η -almost surely, so that the model has a **unique Gibbs measure**.
- A natural **quantitative parameter** is $m_L := \mathbb{E}(\langle \sigma_0 \rangle_{\Lambda_L}^+)$ where $\langle \cdot \rangle_{\Lambda_L}^+$ denotes the thermal expectation in $\{-L, \dots, L\}^2$ with $+1$ boundary conditions.
- A bound of the form $m_L \leq \exp(-c(\lambda, T)L)$ is relatively simple for large disorder strength λ or high temperatures T , so interested in **small λ and low temperature**.
- **Results**: In dimension $d = 2$: $m_L \leq \frac{C(\lambda)}{\sqrt{\log \log L}}$ (Chatterjee 2017), $m_L \leq \frac{C(\lambda)}{L^{c(\lambda)}}$ (Aizenman-P. 2018) and finally

$$m_L \leq C(\lambda) \exp\left(-\frac{L}{\ell(\lambda)}\right)$$

proved at zero temperature by Ding-Xia 2019 and then at positive temperature by Ding-Xia 2019 and Aizenman-Harel-P. 2019.

- Still **open** to determine **correlation length** $\ell(\lambda)$ for small λ . Proof implies $\ell(\lambda) \leq e^{1/\lambda^2}$ (Bar-Nir 2022). Ding-Wirth (2020): Correlation length $= e^{\Theta(\lambda^{-\frac{4}{3}})}$ in another sense.

Random-field Ising and Potts models

- Dimension $d \geq 3$, weak disorder (small λ): [Imbrie 1985](#) (zero temperature) and [Bricmont-Kupiainen 1988](#) (all temperatures) established long-range order in the random-field Ising model. A shorter argument was given recently by [Ding-Zhuang \(2021\)](#), also extending the result to the random-field Potts model.
- [Ding-Liu-Xia \(2022\)](#), making use of [Ding-Song-Sun \(2023\)](#), extend the long-range order result to all temperatures lower than the critical temperature of non-disordered Ising model. [Ding-Huang-Xia \(2023\)](#) investigate the critical scaling for the disorder at the critical temperature of the non-disordered Ising model.
- [Rigas \(2022\)](#) extended part of the correlation length result of [Ding-Wirth](#) to the random-field Potts model.

Quantitative results

- The other models discussed (Potts, spin-glasses, spin $O(n)$) **do not share the monotonicity properties** of the random-field Ising model and the proof techniques break down for them. Indeed, even the choice of which quantity to bound is non-obvious since it is unclear which boundary conditions τ maximize or minimize the average $\langle f(\mathcal{T}_v(\sigma)) \rangle_{\Lambda_L^2}^\tau$ and, indeed, it may be that these boundary conditions depend on the disorder η and on L and v . We obtain the following results.
- Theorem** (Dario-Harel-P 2020+): For each **two-dimensional** disordered lattice system of the type described above, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, there exists $C > 0$ so that for all $L \geq 2$,

$$\mathbb{E} \left(\sup_{\tau_1, \tau_2: \mathbb{Z}^2 \rightarrow S} \left\| \frac{1}{L^2} \sum_{v \in \Lambda_L^2} \langle f(\mathcal{T}_v(\sigma)) \rangle_{\Lambda_L^2}^{\tau_1} - \langle f(\mathcal{T}_v(\sigma)) \rangle_{\Lambda_L^2}^{\tau_2} \right\| \right) \leq \frac{C}{(\log \log L)^{\frac{1}{4}}}$$

For the **d -dimensional random-field spin $O(n)$ model with $n \geq 2$** , at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, there exists $C > 0$ so that for all $L \geq 2$,

$$\mathbb{E} \left(\sup_{\tau: \mathbb{Z}^d \rightarrow S} \left\| \frac{1}{L^d} \sum_{v \in \Lambda_L^d} \langle \sigma_v \rangle_{\Lambda_L^d}^\tau \right\| \right) \leq C \begin{cases} L^{-\frac{1}{3}} & d = 2 \\ L^{-\frac{1}{5}} & d = 3 \\ (\log \log L)^{-\frac{1}{2}} & d = 4 \end{cases}$$

Uniqueness problem

- **Conjecture:** For a disordered lattice system with compact state space (as discussed above) in dimension $d = 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, it holds that η -almost surely, for all vertices $v \in \mathbb{Z}^2$, the value of

$$\langle f(\mathcal{T}_v(\sigma)) \rangle_\mu$$

is the same for all Gibbs measures μ of the η -disordered system.

- The conjecture is equivalent to the following **finite-volume statement**:

$$\lim_{L \rightarrow \infty} \sup_{\tau_1, \tau_2: \mathbb{Z}^2 \rightarrow S} \left\| \langle f(\sigma) \rangle_{\Lambda_L^2}^{\tau_1} - \langle f(\sigma) \rangle_{\Lambda_L^2}^{\tau_2} \right\| = 0, \quad \eta\text{-almost surely}$$

- The value of $\mathcal{T}_v(\sigma)$ itself need not be unique in general systems. For instance, a global sign flip applied to σ in a spin glass system (with Hamiltonian $H^\eta(\sigma) = -\lambda \sum_{u \sim v} \eta_{u,v} \sigma_u \sigma_v$) takes one Gibbs measure to another.
- Applied to two-dimensional spin glasses at zero temperature, the conjecture implies the **conjecture that the spin glass system has a unique ground-state pair**.

Partial uniqueness result

- Due to the disorder in the systems considered, it does not make sense to consider **translation-invariant Gibbs measures**. Instead, the following notion of translation-covariant Gibbs measures has been proposed.
- A measurable map ρ from the disorder variables η to the Gibbs measures of the η -disordered system is called a **translation-covariant Gibbs measure** if

$$\rho(\mathcal{T}_v(\eta)) = \mathcal{T}_v(\rho(\eta))$$

for all vertices $v \in \mathbb{Z}^d$ (the translation \mathcal{T}_v naturally extends to Gibbs measures).

- **Compactness arguments** (Aizenman-Wehr, Newman-Stein) show that translation-covariant Gibbs measures always exist for the disordered systems considered above (as barycenters of **translation-covariant metastates**).
- **Theorem**: For a disordered lattice system with compact state space (as discussed above) in **dimension $d = 2$** , at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, it holds that η -almost surely, for all vertices $v \in \mathbb{Z}^2$, the value of

$$\langle f(\mathcal{T}_v(\sigma)) \rangle_{\rho(\eta)}$$

is the same for all **translation-covariant Gibbs measures ρ** .

- **Corollary**: For the two-dimensional spin glass model at **zero temperature**, if there **exists** a translation-covariant **extremal** Gibbs measure then there is a **unique translation-covariant Gibbs measure** up to a global sign flip.

Proof sketch for compact state space

- **Theorem recalled:** For the above disordered systems with compact state space in two dimensions, at $0 \leq T < \infty$ and $\lambda > 0$, there exists $C > 0$ so that for all $L \geq 2$,

$$\mathbb{E} \left(\sup_{\tau_1, \tau_2: \mathbb{Z}^2 \rightarrow S} \left\| \frac{1}{L^2} \sum_{v \in \Lambda_L^2} \langle f(\mathcal{T}_v(\sigma)) \rangle_{\Lambda_L^2}^{\tau_1} - \langle f(\mathcal{T}_v(\sigma)) \rangle_{\Lambda_L^2}^{\tau_2} \right\| \right) \leq \frac{C}{(\log \log L)^{\frac{1}{4}}}$$

- To simplify, assume $f(\sigma) = f(\sigma_0) \in \mathbb{R}$ and fix $T > 0$. Write $Z_{T, \Lambda, \tau}^\eta$ for the **partition function** at temperature T , in a finite $\Lambda \subset \mathbb{Z}^2$ and with boundary conditions τ . Thus

$$Z_{T, \Lambda, \tau}^\eta := \int e^{-\frac{1}{T} H_\Lambda^\eta(\sigma)} \prod_{v \in \Lambda} d\kappa(\sigma_v) \prod_{v \in \Lambda^c} \delta_{\tau_v}(\sigma_v)$$

with $H_\Lambda^\eta(\sigma)$ the terms in the Hamiltonian $H^\eta(\sigma) = H(\sigma) - \lambda \sum_v \eta_v f(\mathcal{T}_v(\sigma))$ depending on the spins in Λ . Let $F_\Lambda^\eta(\tau) := \frac{T}{|\Lambda|} \log Z_{T, \Lambda, \tau}^\eta$ be minus the free energy.

- **Standard facts:** 1) $F_\Lambda^\eta(\tau)$ is a **convex** function of η .
- 2) For each Λ : $\sup_{\tau_1, \tau_2} |F_\Lambda^\eta(\tau_1) - F_\Lambda^\eta(\tau_2)| \leq \frac{C|\partial\Lambda|}{|\Lambda|}$.
- 3) Write $\eta = (\hat{\eta}_\Lambda, \eta_\Lambda^\perp)$ where $\hat{\eta}_\Lambda := \frac{1}{|\Lambda|} \sum_{v \in \Lambda} \eta_v$ and $\eta_{\Lambda, v}^\perp := \eta_v - \hat{\eta}_\Lambda$. Then

$$\frac{\partial}{\partial \hat{\eta}_\Lambda} F_\Lambda^{(\hat{\eta}_\Lambda, \eta_\Lambda^\perp)}(\tau) = \frac{\lambda}{|\Lambda|} \sum_v \langle f(\mathcal{T}_v(\sigma)) \rangle_\Lambda^\tau, \text{ with the sum over terms involving spins in } \Lambda$$

Proof sketch II

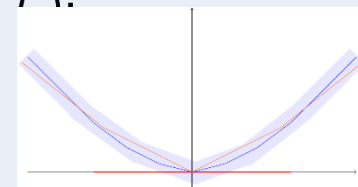
- **Lemma:** Let Λ satisfy $|\partial\Lambda| \leq C\sqrt{|\Lambda|}$. Then for each $\delta > 0$,

$$\mathbb{P} \left(\sup_{\tau_1, \tau_2: \mathbb{Z}^d \rightarrow S} \left| \frac{\lambda}{|\Lambda|} \sum_v f \left(\mathcal{T}_v \left(\sigma_{\Lambda, \tau_1}^\eta \right) \right) - f \left(\mathcal{T}_v \left(\sigma_{\Lambda, \tau_2}^\eta \right) \right) \right| < 2\delta \right) \geq \exp \left(-\frac{C\lambda^2}{\delta^4} \right)$$

- **Proof sketch: Claim:** Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a **convex 1-Lipschitz** function. Set $N_r(g) := \{ h: \mathbb{R} \rightarrow \mathbb{R} \text{ convex 1-Lipschitz} \mid \|h - g\|_\infty \leq r \}$.

Then for each $r, \delta > 0$.

$$\text{Leb}(\{ x \in \mathbb{R} \mid \exists h \in N_r(f), |h'(x) - g'(x)| \geq \delta \}) \leq \frac{Cr}{\delta^2}$$



- Fix $\tau_0: \mathbb{Z}^2 \rightarrow S$ and let $g(x) := F_\Lambda^{(x, \eta_\Lambda^\perp)}(\tau_0)$. Then for all τ , $F_\Lambda^{(\cdot, \eta_\Lambda^\perp)}(\tau) \in N_{\frac{C|\partial\Lambda|}{|\Lambda|}}(g)$.

Thus, the Claim implies that

$$\text{Leb} \left(\left\{ x \in \mathbb{R} \mid \exists \tau: \mathbb{Z}^2 \rightarrow S, \left| \frac{\partial}{\partial \hat{\eta}_\Lambda} g_\Lambda^{(x, \eta_\Lambda^\perp)}(\tau) - \frac{\partial}{\partial \hat{\eta}_\Lambda} g_\Lambda^{(x, \eta_\Lambda^\perp)}(\tau_0) \right| \geq \delta \right\} \right) \leq \frac{C\lambda|\partial\Lambda|}{|\Lambda|\delta^2} \leq \frac{C\lambda}{\sqrt{|\Lambda|}\delta^2}$$

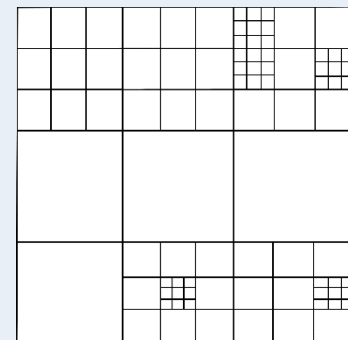
- Since $\hat{\eta}_\Lambda := \frac{1}{|\Lambda|} \sum_{v \in \Lambda} \eta_v$ is **Gaussian** with standard deviation $\frac{1}{\sqrt{|\Lambda|}}$ we conclude that

$$\mathbb{P} \left(\sup_{\tau: \mathbb{Z}^d \rightarrow S} \left| \frac{\lambda}{|\Lambda|} \sum_v f \left(\mathcal{T}_v \left(\sigma_{\Lambda, \tau}^\eta \right) \right) - f \left(\mathcal{T}_v \left(\sigma_{\Lambda, \tau_0}^\eta \right) \right) \right| < \delta \right) \geq \exp \left(-\frac{C\lambda^2}{\delta^4} \right)$$

Proof sketch III

- Let $L \geq 2$. Call a set $\Lambda' \subset \Lambda_L$ **ϵ -fluctuative** if

$$\sup_{\tau_1, \tau_2: \mathbb{Z}^d \rightarrow S} \left| \frac{\lambda}{|\Lambda'|} \sum_v f\left(\mathcal{T}_v\left(\sigma_{\Lambda', \tau_1}^\eta\right)\right) - f\left(\mathcal{T}_v\left(\sigma_{\Lambda', \tau_2}^\eta\right)\right) \right| < \epsilon$$



- Perform a **fractal percolation**: Set $\delta := \frac{C\sqrt{\lambda}}{(\log \log L)^{\frac{1}{4}}}$ and $k = C\lambda/\delta$.

Partition Λ_L into k squares. Then partition each of these into k squares and so on until reaching squares of constant size. A square in this recursive partition is **taken** if it is 4δ -fluctuative and the squares containing it are not 4δ -fluctuative.

- Define $B := \{v \in \Lambda_L \mid v \text{ is not in a taken square}\}$. Then

$$\sup_{\tau_1, \tau_2: \mathbb{Z}^d \rightarrow S} \left| \frac{\lambda}{|\Lambda_L|} \sum_v f\left(\mathcal{T}_v\left(\sigma_{\Lambda_L, \tau_1}^\eta\right)\right) - f\left(\mathcal{T}_v\left(\sigma_{\Lambda_L, \tau_2}^\eta\right)\right) \right| \leq 4\delta + \frac{C|B|}{|\Lambda_L|}$$

- It remains to show that $\mathbb{P}(v \in B) \leq \delta$. Write $\Lambda_0(v) \supset \Lambda_1(v) \supset \Lambda_2(v) \supset \dots$ for the partition squares containing v . Since $|\Lambda_{\ell+1}(v)| \leq c\delta|\Lambda_\ell(v)|/\lambda$, one concludes that

$$\{v \in B\} \subset \bigcap_{\ell} \{\Lambda_\ell(v) \setminus \Lambda_{\ell+1}(v) \text{ is not } 2\delta\text{-fluctuative}\}$$

- The events in the intersection are **independent** since the annuli are disjoint.

The previous lemma bounds their probabilities, concluding the proof.

Non-compact case:

Random-field random surfaces

- We now discuss the effect of disorder on systems with **non-compact state space**. Our focus is on **random surface** models.
- Let $(\eta_v)_{v \in \mathbb{Z}^d}$ be independent standard Gaussian random variables.
- A real-valued **random-field random surface** is the model on $\phi: \mathbb{Z}^d \rightarrow \mathbb{R}$ with Hamiltonian

$$H^\eta(\phi) = \sum_{u \sim v} V(\phi_u - \phi_v) - \lambda \sum_v \eta_v \phi_v$$

where $V: \mathbb{R} \rightarrow \mathbb{R}$ is a measurable even function termed the **potential**.

The case $V(x) = x^2$ is the real-valued random-field **Gaussian free field**.

- We also study the **integer-valued** random-field Gaussian free field which has the same Hamiltonian as above with $V(x) = x^2$ but restricts to $\phi: \mathbb{Z}^d \rightarrow \mathbb{Z}$.
- Our goal the **localization/delocalization** properties of these disordered surfaces.
- **Without disorder**: the **gradient** of these surfaces localizes in all dimensions $d \geq 1$. On Λ_L^d , **real-valued** surfaces delocalize with variance L when $d = 1$ and with variance $\log L$ when $d = 2$ while staying localized for $d \geq 3$. The integer-valued GFF behaves similarly except for a **roughening transition** when $d = 2$, from localized to logarithmic delocalization as the temperature increases.

Random-field random surfaces: results

- **Theorem** (Dario-Harel-P 2020+): Consider the **real-valued** random-field random surfaces above at all temperatures $0 \leq T < \infty$ and all disorder strengths $\lambda > 0$ on Λ_L^d with zero boundary conditions. Assume $0 < c_- \leq V'' \leq c_+ < \infty$. Then

- Discrete Gradient: $\mathbb{E} \left(\left\langle \frac{1}{L^d} \sum_{\{u,v\} \in E(\Lambda_L^d)} (\phi_u - \phi_v)^2 \right\rangle \right) \approx \begin{cases} L & d = 1 \\ \log L & d = 2 \\ 1 & d \geq 3 \end{cases}$

- Height fluctuations: $\mathbb{E}(\langle \phi_0 \rangle^2) \approx \begin{cases} L^{4-d} & d = 1, 2, 3 \\ \log L & d = 4 \\ 1 & d \geq 5 \end{cases}$

- **Theorem** (Dario-Harel-P 2020+): The **integer-valued** random-field Gaussian free field, at all temperatures $0 \leq T < \infty$ and disorder strengths $\lambda > 0$, satisfies the **gradient estimate above**, and, when $d = 1, 2$, satisfies

$$\mathbb{E} \left(\left\langle \frac{1}{L^d} \sum_{v \in \Lambda_L^d} \phi_v^2 \right\rangle \right) \approx L^{4-d}$$

Additionally, this expectation is **bounded in L in dimensions $d \geq 3$** at low temperatures and **small disorder strength $\lambda > 0$** .

Random-field random surfaces: previous results

- **Bovier-Külske** studied a random field Solid-On-Solid model in which the disorder enters differently from the way it is introduced here. They proved a certain form of delocalization in two dimensions (**Bovier-Külske 1996**) and localization in three and higher dimensions (**Bovier-Külske 1994**).
- **Külske and Orlandi 2006** prove that for **all deterministic fields η** , a random surface with field η will delocalize with **at least logarithmic variance** in two dimensions, when the potential V satisfies $\sup V(x) < \infty$.
- **Van Enter and Külske 2008** proved a form of delocalization for the gradients of the random-field random surface for a wide class of potentials in two dimensions. The result is non-quantitative.
They further proved a lower bound on the rate of correlation decay for gradient Gibbs measures, when they exist, in three dimensions.
- **Cotar and Külske** proved the existence of translation-covariant gradient Gibbs measures for random-field random surfaces in dimensions $d \geq 3$ (**Cotar and Külske 2012**) and their uniqueness for each given expected tilt (**Cotar and Külske 2015**), for a large class of potentials.
- Later results: **Dario 2023** (thermodynamic limit), **Sakagawa 2023** (maximum).

Open questions

- For disordered systems with compact state space, improve the bounds on

$$\mathbb{E} \left(\sup_{\tau_1, \tau_2: \mathbb{Z}^d \rightarrow S} \left\| \frac{1}{L^d} \sum_{v \in \Lambda_L^d} \langle f(\mathcal{T}_v(\sigma)) \rangle_{\Lambda_L^d}^{\tau_1} - \langle f(\mathcal{T}_v(\sigma)) \rangle_{\Lambda_L^d}^{\tau_2} \right\| \right)$$

If the sum is performed over a concentric box of half the size, does it decay **exponentially fast** with L in two dimensions at all T and $\lambda > 0$?

- Uniqueness conjecture:** For two-dimensional disordered systems, for each $v \in \mathbb{Z}^2$, η -almost surely, the value of $\langle f(\mathcal{T}_v(\sigma)) \rangle_\mu$ is the same for all Gibbs measures μ .
- Is there a **Berezinskii-Kosterlitz-Thouless** type transition as the disorder strength lowers (i.e., transition from **exponential to power-law decay**) for the random-field spin $O(n)$ models with $n = 2$ in dimensions $d = 3$ or $d = 4$? What about $n \geq 3$?
- What is the **localization/delocalization** behavior of the **integer-valued** random-field Gaussian free field in dimensions $d \geq 3$ at high disorder strength λ ?
Conjecture: Delocalization in dimension $d = 3$ and localization when $d \geq 5$.
 Thus we conjecture a **roughening transition** in the disorder strength for $d = 3$.